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# A numerical study of the roll-up of a finite vortex sheet

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A point-vortex representation is used to study numerically the evolution of an initially plane vortex sheet. By introducing a tip vortex to represent the tightly rolled portion of the vortex sheet, the chaotic motion which was a feature of some earlier studies is eliminated and the details of the outer portion of the spiral are calculated. The rate of rolling up is calculated and is shown to be governed by the analytically predicted similarity law of Kaden during the initial stages of the rolling up. The calculations are continued until 99 % of the vorticity has been rolled up, at which stage the spiral displays a marked ellipticity.

## 1. Introduction

At time $t = 0$ a vortex sheet of infinite length whose constituent vortex lines are parallel to the $z$ axis intersects the $x$, $y$ plane in the strip $y = 0$, $-a \leqslant x \leqslant a$. At this instant the vortex-sheet strength $\omega(x)$ is given by

$$\omega(x) = 2Ux(a^2 - x^2)^{-\frac{1}{2}}, \tag{1.1}$$

where $U$ is the velocity with which the sheet is instantaneously moving in the $y$-decreasing direction. The rectangular axes $Oxyz$ are chosen so that the fluid at infinity is at rest.

The fluid is incompressible and if viscosity is neglected the problem of determining the evolution of the vortex sheet appears to be straightforward. The flow field is two-dimensional and the problem can be discretized by subdividing the strip and replacing each subdivision by a point vortex at its vorticity centroid. The evolution of the sheet is followed by integrating forwards in time the equations which give the velocity of each point vortex as a function of the position of the others.

This calculation was first attempted by Westwater (1935), who chose his subdivisions so that the strengths of the point vortices were of equal magnitude. A spiral structure for the vortex sheet near the tips emerged and the rate of rolling up of the vortex sheet could be estimated.

Unfortunately, attempts to reproduce Westwater's results by Takami (1964) and Moore (1971, subsequently referred to as I) have not been successful. The vortices moved chaotically, no spiral structure emerged and the rate of roll-up could not be estimated.

Now the analytical work of Kaden (1931) makes it clear that smooth spirals form at the ends of the vortex sheet, so that one has to try to understand why this spiral structure was not obtained by Takami or in I. There are two possibilities.

The first is that the exact solution of the discrete system converges to the solution of the equations governing the continuous sheet as the number of point vortices tends to infinity. The chaotic motion would then be due to a failure of the numerical method to integrate the discrete system correctly. This failure could spring from either the use of too crude an integration method – for example, the time step might be too large – or because the discrete equations themselves are numerically unstable in the sense that small perturbations, always present in a numerical calculation, are rapidly amplified.

The second possibility is that the exact solution of the discrete system does not converge to the solution of the equations governing the continuous sheet as the number of point vortices tends to infinity.

In I an attempt was made to rule out the first possibility. Sixteen-figure arithmetic was used in conjunction with the fourth-order Runge–Kutta integration scheme. Provided that the time step is sufficiently small this method is stable and it was possible to follow 40 complete revolutions of a vortex pair with a fractional error of only about $10^{-9}$ in the co-ordinates. In repeating Westwater's calculations, the time step was chosen to be much smaller than the orbital period of the two closest vortices. Given that this condition was satisfied, the solution was insensitive to the actual choice of time step. Chaotic motion quickly developed, just as reported by Takami, and fair agreement between the actual vortex positions was noted, confirming that the discrete equations were being correctly integrated. Moreover, if at some instant the vortex strengths were instantaneously reversed, the chaotic state would unscramble and the vortices would return to their original positions.

Thus it was concluded in I that it was the discretization itself which was responsible and this was confirmed by showing that increasing the number of vortices made matters worse. The reason for Westwater's success in obtaining a spiral structure was not found. However, an interesting suggestion has been made by Chorin & Bernard (1972), which is that a combination of Euler integration and large time steps, as employed by Westwater, will prevent chaotic motion, and this is borne out by the recent work of Clements & Maull (1973), who used a time step comparable with the orbital period of the closest vortices. Chorin & Bernard do not advance an explanation of the inhibition of chaotic motion, but it is possibly due to the fact that such an integration procedure quickly increases the separations between the vortices near the ends of the sheet to much greater than their true values. This in turn will tend to suppress the orbiting motion of vortex pairs which was a prominent feature of the chaotic motion found in I. Alternatively, one can regard the augmented separations as fulfilling a function similar to Chorin & Bernard's cut-off, described in detail below.

This method of dealing with the problem cannot be regarded as satisfactory because sizeable errors are introduced. As was pointed out by Crow (1965), Euler integration introduces a cumulative error when applied to vortex motion. For example, one can show, by studying the finite-difference equations analytically, that, when applied to a pair of vortices of equal strengths, Euler integration leads to a vortex separation which continually increases instead of remaining constant. Moreover, if the time step is comparable with the orbital period, large errors are quickly introduced. It is clear that Clements & Maull's results (whilst probably adequate for their purpose) are not accurate in the spiral region because the authors found that a change of time step caused a change in the shape of the spiral comparable with its radius.

The failure of the discretization must be a consequence of the nature of the motion near the tips of the rolling-up vortex sheet. One explanation, which in essence has been proposed by several authors, arises from the spiral shape of the sheet. If the distance between turns is much less than the typical arc distance between the constituent vortices then there will be instants when vortices on neighbouring turns will come very close together. This will lead to a spuriously large interaction between this pair which might disrupt the orderly evolution of the system.

The first study of roll-up which incorporated a recipe for suppressing chaotic motion based on this diagnosis was that of Nielsen & Schwind (1971), who combined two vortices which came closer to each other than a critical distance (which was comparable with the initial separation of neighbouring vortices) into a single vortex. This vortex was at the centroid of the pair and had a strength equal to the total strength of the pair. Smooth roll-up was reported, but no details were given. A different recipe was employed by Chorin & Bernard (1972). They modified the velocity field of each point vortex in such a way that it had its correct value outside a cut-off radius, but remained bounded inside this radius. The cut-off radius was chosen to be of the order of the separation of nearest neighbours in the initial state. The authors regard their modification as being equivalent to the introduction of a fictitious viscosity, because the vorticity of each point vortex is spread out in a circle of radius equal to the cut-off radius.

To describe their results in more detail, it is convenient to introduce a dimensionless time $t^*$ defined by

$$t^* = Ut/a; \tag{1.2}$$

clearly $a/U$ is the natural time scale of the system. Chorin & Bernard found smooth spirals for $t^* \leqslant 1{\cdot}0$, though at $t^* = 1{\cdot}0$ their results possess unrealistic features. In particular, the vortices, which were originally equidistant, do not display steadily increasing separation as the spiral is entered, whereas Kaden's analysis suggests that the sheet is increasingly stretched towards the centre of the spiral. At $t^* = 6{\cdot}5$ the spiral structure has disappeared and the motion appears chaotic.

A similar cut-off has been employed by Kuwahara & Takami (1973), who used instead the velocity field of a diffusing line vortex. Smooth roll-up was obtained up to $t^* = 0{\cdot}5$, the largest value considered. However, some dependence of the results on the magnitude of the coefficient of viscosity used is evident, and it is not clear which value is most appropriate.

A second possible explanation of the failure of the discrete representation arises out of an examination of Kaden's results. When $t^* \ll 1$ the spirals at the ends of the vortex sheet evolve independently and Kaden showed analytically that the vortex sheet at either tip has the form of a spiral whose polar equation is

$$r \propto (t^*/\theta)^{\frac{2}{3}}. \tag{1.3}$$

Thus any attempt to replace the sheet by a finite number of point vortices will cease to be adequate sufficiently near the centre of the spiral, since the spiral has an infinite number of turns.

This is obvious and Westwater pointed out that no detail of the inner portion of the spiral could be found by his method. What is less obvious is that the failure of the discretization is not local. There is no reason to assume that the vortices representing the inner spiral portion of the sheet maintain their correct positions on the vortex sheet but, even if they did, the vortices in the outer part of the sheet would not experience the true velocity field induced by the inner spiral. This velocity field varies smoothly with time and is almost axisymmetric, whereas the velocity field due to these inner point vortices is irregularly fluctuating and is non-axisymmetric. Thus the outer vortices, which one might have hoped would represent correctly the outer part of the sheet, respond by themselves starting to move irregularly. Thus chaotic motion spreads to ruin the calculation, the long-range nature of the coupling between individual vortices facilitating the process.

If this diagnosis is correct it must be possible to remove chaotic motion by representing the inner turns in a way which removes the spurious fluctuations.

In § 2 a method of preventing chaotic motion which springs from the second diagnosis is described and the results obtained are discussed in § 3.

## 2. The numerical method

The velocity field of the inner portion of the spiral is very nearly axisymmetric and, instead of trying to represent this by a finite number of point vortices scattered about on the spiral, it is better to represent it by a single vortex at the centre of the spiral. This representation of the inner portion of a spiral vortex sheet was used by Smith (1968) in his study of the formation of the leading-edge vortex over a delta wing.

This suggests the following approach. One can suppose that the velocity field of a turn of the spiral is adequately represented by $N$ or more point vortices, where the integer $N$ has to be determined by trial and error. If a turn proves to contain less than $N$ point vortices, one gives up the attempt to describe it individually, but represents instead the net effect of all such turns by a single vortex of appropriate strength at the centre of the spiral. A practical method of accomplishing this is described below.

The vortex sheet is represented by $2M$ point vortices whose motion can be followed numerically; the integration method used and the ways in which it was tested are described briefly above and in more detail in I. The co-ordinates $(x(i), y(i))$ of the $i$th point vortex with respect to the fixed axes defined in § 1 are denoted by a two-dimensional vector $\mathbf{x}(i)$, where $i$ runs from 1 to $2M$ and where in the initial configuration $i$ increasing means $x(i)$ increasing. The vortex whose co-ordinate is $\mathbf{x}(1)$ is taken to represent the net effect of the inner portion of the spiral which forms at the left-hand tip of the sheet. The angle $\theta$ between the vectors $\mathbf{x}(2)$–$\mathbf{x}(3)$ and $\mathbf{x}(3)$–$\mathbf{x}(4)$ is examined at each time step as the integration proceeds and when this angle exceeds $\theta_c = 360°/N$, $\mathbf{x}(2)$ and $\mathbf{x}(1)$ are combined to form a new tip vortex, with concomitant changes at the right-hand end of the sheet.† The new tip vortex is placed at the centroid of $\mathbf{x}(1)$ and $\mathbf{x}(2)$ and has circulation equal to the combined circulations of the original pair.

The calculation then proceeds, amalgamation taking place whenever $\theta > \theta_c$. In effect, vortices which try to get too far into the inner spiral are absorbed into the tip vortex, whereas the outer turns of the spiral have at least $N$ vortices on each.

In practice it was found that $\theta_c = 90°$, corresponding to 4 vortices per turn, was adequate to prevent chaotic motion, whereas $\theta_c = 120°$, corresponding to 3 vortices per turn, was not.

For obtaining details of the early stages of roll-up Westwater's discretization was used. As discussed in I, this places a restriction on the time step $dt^*$,

$$dt^* \ll \pi^2 M^{-3}. \tag{2.1}$$

However, for studying the later stages of the motion a discretization in which the subdivisions are of equal length is preferable. This mode of subdivision gives better definition of the middle portion of the sheet and, because the minimum orbital period of any pair of constituent vortices is increased, enables larger time steps to be used. The orbital period of the tip vortex and its nearest neighbour is less than the circulation period at a radius equal to their separation of a single vortex containing all the circulation. This circulation period is $4\pi^2 r^2/\Gamma$, where $r$ is the separation of the tip vortex and its nearest neighbour and $\Gamma$ is the total circulation. But for the distribution (1.1) $\Gamma = 2Ua$ and $r \sim a/M$.‡ Thus one must ensure that $dt^*$ satisfies

$$dt^* \ll 2\pi^2 M^{-2}, \tag{2.2}$$

a less restrictive condition than (2.1).

## 3. Discussion and results

It is not easy to estimate the error introduced by the amalgamation process described in § 2. That some error is inevitable is clear from an examination of its effects on the invariants of the motion. It can be shown that, if $\{K_i\}$ are the vortex strengths,

$$\sum_{i=1}^{M} K_i x(i)$$

and

$$\sum_{i \neq j}\sum K_i K_j \log|\mathbf{x}(i) - \mathbf{x}(j)|$$

† In many of the calculations symmetry about $Oy$ was imposed. This proved not to affect the results, which were very accurately symmetric anyway, and saved computing time.

‡ It is not exactly $a/M$, because the vortices are at the vorticity centroids, not the geometric centres of the subdivisions.

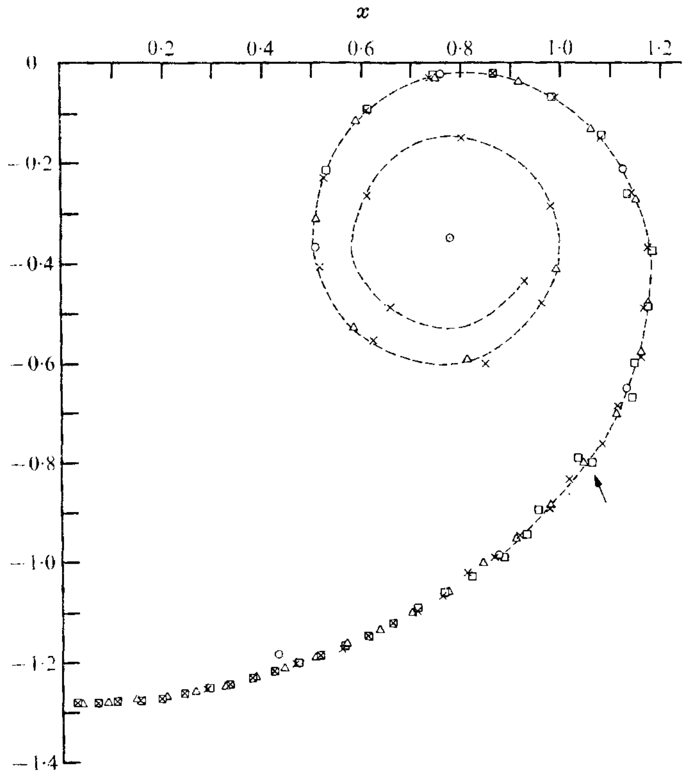


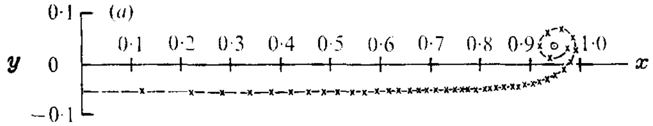

FIGURE 1. Comparison of spiral shape at $t^* = 2{\cdot}0$. $\times$, equispaced, $M = 60$, $\theta_c = 90°$, $dt^* = 2\times10^{-3}$; $\square$, equispaced, $M = 60$, $\theta_c = 30°$, $dt^* = 2\times10^{-3}$; $\bigcirc$, equal strengths, $M = 30$, $\theta_c = 90°$, $dt^* = 5\times10^{-4}$; $\triangle$, equispaced, $M = 45$, $\theta_c = 70°$, $dt^* = 2{\cdot}5\times10^{-3}$. The tip vortex is denoted by $\odot$ and its position was virtually the same in all cases. The dashed line joins the vortices in their original order. The arrow marks the position of the instability.

are invariants of the motion. The first but not the second is preserved by the amalgamation process.†

The best evidence that this error is not serious is provided by the consistency of calculations with different values of $\theta_c$ and a comparison of results from calculations with three different values of $\theta_c$ is shown in figure 1. Evidently the details of the outer part of the spiral are not sensitive to the choice of $\theta_c$. As might have been anticipated, the run with largest $M$ and greatest $\theta_c$ gives the most information about the shape of the spiral. There are indications of an instability, which it is plausible to identify with Kelvin–Helmholtz instability, at the position marked by the arrow. The vortices in one run can be seen to be departing from the position of the sheet as defined by the other runs. The role of Kelvin–Helmholtz instability is discussed further below.

Figures 2, 3 and 4 give the shape of the spiral at various times, the right-hand half of the sheet being displayed. The axes are those defined in § 1, though for

† Both invariants were computed and found to be sensibly constant between amalgamations. This provides a check on the accuracy of the numerical integration.



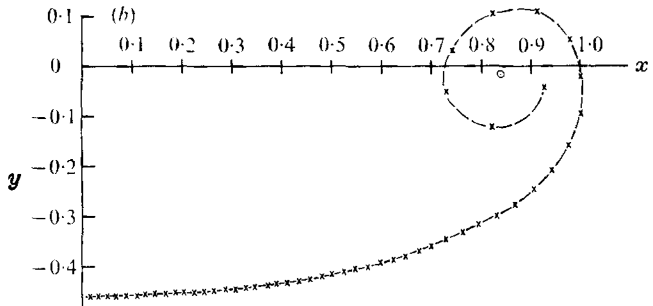




FIGURE 2. Spiral shapes at different times. $\theta_c = 90°$, $M = 60$. (a) $t^* = 0{\cdot}05$, $dt^* = 2\times10^{-5}$, vortices of equal strength. (b) $t^* = 0{\cdot}5$, $dt^* = 2\times10^{-3}$, vortices initially equispaced.

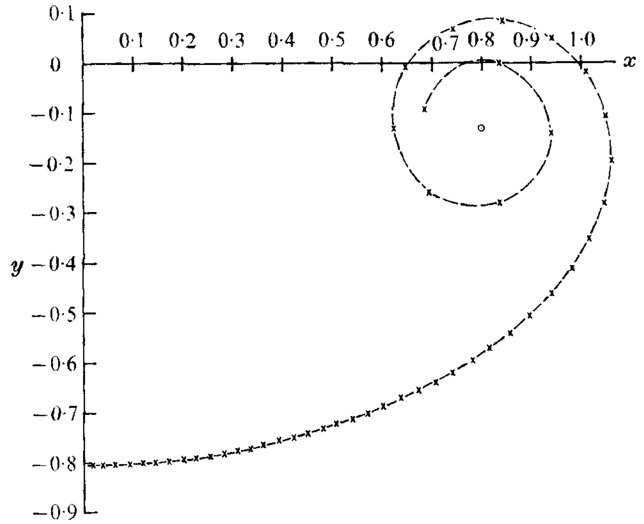



FIGURE 3. Spiral shape at $t^* = 1{\cdot}0$. $\theta_c = 90°$, $M = 60$, $dt^* = 2\times10^{-3}$, vortices initially equispaced.

convenience $x/a$ and $y/a$ are plotted rather than $x$ and $y$. The centre of the sheet descends with a velocity initially close to $U$, though it subsequently decelerates. The tip vortex descends much more slowly, having initially ascended slightly. It is worth recalling that if the sheet rolled up into two line vortices these would have $x$ co-ordinates $\pm\frac{1}{4}\pi a$ or $\pm 0{\cdot}785a$ and would descend with speed

$$2U/\pi^2 \doteqdot 0{\cdot}2U.$$

The results are consistent with these rough estimates.

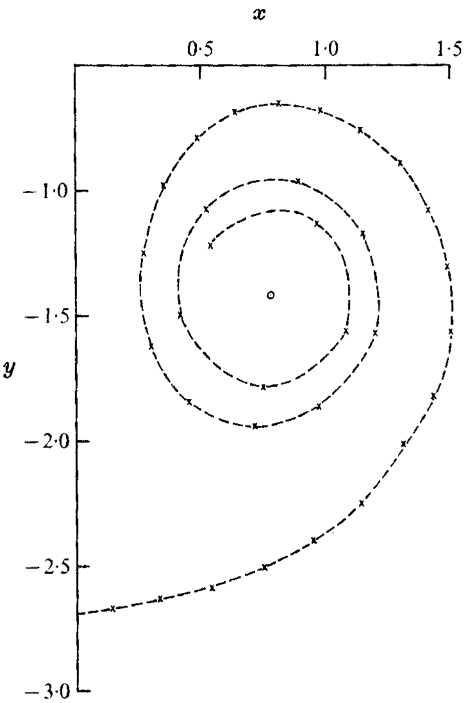



FIGURE 4. Spiral shape at $t^* = 7 \cdot 0$. $\theta_c = 90°$, $M = 60$, $dt^* = 2 \times 10^{-3}$, vortices initially equispaced.

Several other features of these results may be noted.

(*a*) The stretching of the vortex sheet as it enters the spiral is made apparent by the separation of the constituent vortices, which were (except for the run which led to figure 2(*a*)) initially equally spaced.

(*b*) The results (figure 3) at $t^* = 1 \cdot 0$ can be compared with those of Chorin & Bernard. The unphysical features noted in their results are not present here, although the agreement between the two calculations is in general quite fair.

(*c*) In contrast to Chorin & Bernard's findings, spiral structure persists until large times. The results (figure 4) for $t^* = 7 \cdot 0$ (the largest time considered) show a tendency for the turns of the spiral to be elliptical rather than circular while the stretching, as evidenced by the vortex spacings, does not increase monotonically. However, it is not clear how accurate the calculations are at such large times and in view of the large amount of computing required (about 15 min on the CDC 6600) it was not possible to support these results by comparing different runs.

(*d*) It is remarkable that there is so little sign of Kelvin–Helmholtz instability, and one must conclude that it is inhibited. The growth rate of instability for a uniform vortex sheet is proportional to the wavelength of the disturbance, so that in a point-vortex representation the growth rate is that corresponding to a wave of length comparable with the distance between neighbouring vortices. (A precise calculation is given in Lamb 1932, p. 225.) Thus, increasing $M$ increases the likelihood of encountering Kelvin–Helmholtz instability and the runs

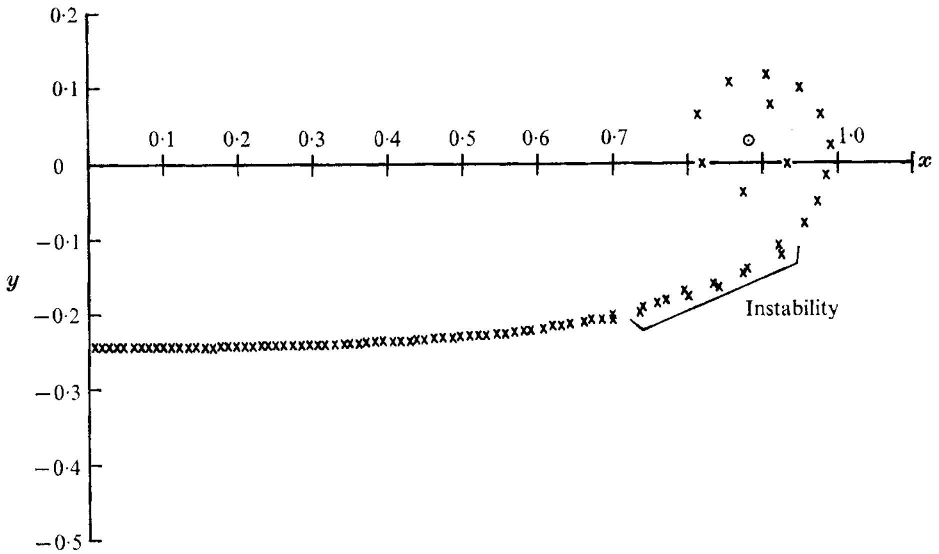



FIGURE 5. Spiral at $t^* = 0{\cdot}25$ with $\theta_c = 90°$, $M = 98$, $dt^* = 10^{-3}$, showing instability.

with $M = 60$ show some signs of this, particularly the run with $\theta_c = 30°$. The instability is not in the rolled-up part of the sheet, where perhaps it is reduced by the rapid stretching, but occurs between the rolled-up and unstretched parts of the sheet. Professor P. G. Saffman has proposed an interesting explanation of the location of the Kelvin–Helmholtz instability. In the initial state of the vortex sheet the vorticity $\omega(s)$ is a monotone increasing function of the arc distance $s$ measured from the centre of the sheet. However, at any subsequent time it is known from Kaden's analysis that $\omega \to 0$ as the centre of the spiral is approached. Thus once the spiral forms $\omega$ must have a maximum at some intermediate value of $s$. Examination of the results of the computations show that, for the times of interest, the maximum value of $\omega(s)$ occurs in the same region as the Kelvin–Helmholtz instability, tending to confirm Professor Saffman's explanation.

The magnitude of the disturbance must reflect the magnitude of the perturbations to which the vortices have been subjected. Now each amalgamation causes an instantaneous small change in the velocity field at the other vortices, so that the amalgamation process itself creates perturbations. The run with $\theta_c = 30$ has involved more amalgamations than the other runs, which may explain why it shows most signs of Kelvin–Helmholtz instability. (I am indebted to Mr J. H. B. Smith for this explanation.)

The conjecture that increasing $M$ promotes instability is supported by the results for a run with $M = 98$ and $\theta_c = 90°$. Instability was evident as early as $t^* = 0{\cdot}25$, and in fact by $t^* = 0{\cdot}5$ chaotic motion had set in. The result for $t^* = 0{\cdot}25$ is shown in figure 5.

The quantity of most interest is the rate of roll-up of the sheet. To make this precise, one can arbitrarily define the rolled-up portion of the sheet to be the portion between the centre of the spiral and the point at which the tangent is

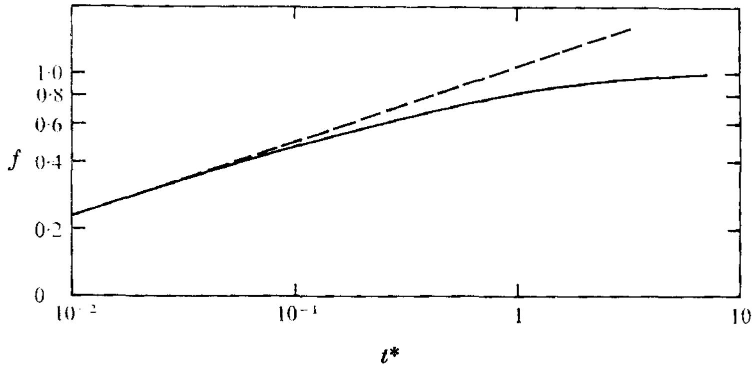



FIGURE 6. The fraction $f$ of vorticity rolled up as a function of $t^*$. The straight dashed line is the similarity law of equation (3.1).

last parallel to $Oy$. Figure 6, which is the principal result of this paper, shows the fraction $f$ of the total vorticity which is rolled up as a function of $t^*$. Clearly roll-up is initially very rapid, 50 % of the vorticity being in the rolled-up portion at $t^* = 0{\cdot}12$.

According to Kaden's analysis, for $t^* \ll 1$ one should have

$$f = gt^{*\frac{1}{3}}, \tag{3.1}$$

where $g$ is a constant. The dashed line on figure 6 is obtained by choosing $g$ so that agreement is obtained at $t^* = 0{\cdot}01$. Evidently (3.1) is an adequate approximation for $t^* < 0{\cdot}1$.

It is worth noting that close approach of vortices on neighbouring turns did not cause trouble in the calculations reported here. Possibly this is because the tip vortex is providing the dominant contribution to the velocities of individual vortices, so that close approach does not lead to disruption.

It is of interest to convert the results obtained for the rate of roll-up into aeronautical terms, relating the two-dimensional unsteady flow to the three-dimensional steady flow behind the wing in the usual approximate way. For an elliptically loaded wing with root circulation $\Gamma_0$, one has

$$U = \Gamma_0/2a, \tag{3.2}$$

where $2a$ is now the wing-span. Thus the characteristic time $a/U$ is $2a^2/\Gamma_0$. For a Boeing 707, $a = 23$ m and a value of $\Gamma_0$ appropriate to cruising at 270 m/s at 10 km altitude can be roughly estimated to be $3{\cdot}5 \times 10^2$ m$^2$/s. Thus the characteristic time is about 3·5 s and roll-up is 50 % complete in about 0·35 s, or just over 2 spans behind the wing, while roll-up is 90 % complete only at about 7 s, or about 1 km behind the aircraft.

This work has benefited greatly from the discussions I have had with Professor P. G. Saffman and from his comments on the first draft.

## REFERENCES


CHORIN, A. J. & BERNARD, P. S. 1972 Discretization of a vortex sheet, with an example of roll-up. (To appear.)

CLEMENTS, R. R. & MAULL, D. J. 1973 The rolling up of a trailing vortex sheet. *Aero. J.* **77**, 46.

CROW, S. 1965 Ph.D. thesis, California Institute of Technology.

KADEN, H. 1931 Aufwicklung einer unstabilen Unstetigkeitsfläche. *Ing. Archiv.* **2**, 140. (English trans. *R.A.E. Library Trans.* no. 403.)

KUWAHARA, K. & TAKAMI, H. 1973 Numerical studies of two-dimensional vortex motion by a system of point vortices. *J. Phys. Soc. Japan*, **34**, 247.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

MOORE, D. W. 1971 The discrete vortex approximation of a vortex sheet. *California Inst. of Tech. Rep.* AFOSR-1084-69.

NIELSEN, J. N. & SCHWIND, R. G. 1971 Decay of a vortex pair behind an aircraft. *Aircraft Wake Turbulence and its Detection*, p. 413. Plenum.

SMITH, J. H. B. 1968 Improved calculations of leading-edge separation from slender delta wings. *Proc. Roy. Soc.* A **306**, 67.

TAKAMI, H. 1964 A numerical experiment with discrete vortex approximation with reference to the rolling up of a vortex sheet. *Dept. of Aeron. & Astron., Stanford University*, SUDAER 202.

WESTWATER, F. L. 1935 Rolling up of the surface of discontinuity behind an aerofoil of finite span. *Aero. Res. Counc. R. & M.* no. 1962.